Stochastic Processes and their Applications 18 (1984) 47–56
North-Holland

SHORT COMMUNICATION

ON THE SIZE OF THE INCREMENTS OF NONSTATIONARY GAUSSIAN PROCESSES

Joaquín ORTEGA*

Instituto Venezolano de Investigaciones Científicas Apartado 1827, Caracas 1010-A, Venezuela, and Universidad Central de Venezuela

Received 8 June 1983
Revised 23 November 1983

Let $\{X(t), t \geq 0\}$ be a centred nonstationary Gaussian process with $EX^2(t) = C_0 t^{2\alpha}$ for some $C_0 > 0$, $0 < \alpha < 1$, and $\beta_T = 1/\sigma(a_T)(2(\log T/a_T + \log\log T)^{1/2}$. In this paper the a.s. asymptotic behaviour of $I(T, a_T)\beta_T$ as $T \to \infty$ is studied where $I(T, a_T) = \sup\{|X(t') - X(t)|: 0 \leq t < t' \leq T, t' - t \leq a_T\}$. The results obtained extend work done by M. Csörgő and P. Révész for the Wiener process.

A.M.S. 1980 Subject Classification: Primary: 60F15; Secondary 60G15, 60G17.

increments of Gaussian processes * increments of Brownian motion * Gaussian processes with stationary increments

1. Introduction

Let $\{X(t), t \geq 0\}$ be a centred Gaussian process with stationary increments and $X(0) = 0$ a.s. We assume that

$$\sigma^2(h) = E(X(t+h) - X(t))^2 = EX^2(h) = C_0 h^{2\alpha} \tag{1}$$

for $0 < \alpha < 1$ and some constant $C_0 > 0$. This condition implies that the process has continuous paths with probability one ([4]).

For $T > 0$ let a_T be a nondecreasing function of T such that $0 < a_T \leq T$ and a_T/T is nonincreasing. We define

$$H(T, a_T) = \sup_{0 \leq t \leq T - a_T} \sup_{0 \leq s \leq a_T} |X(t+s) - X(t)|,$$

$$I(T, a_T) = \sup_{\substack{0 \leq t < t' \leq T \\ t' - t \leq a_T}} |X(t') - X(t)|.$$

Our interest in this work will be centred on the asymptotic behaviour of these objects as $T \to \infty$. In [2] (see also [3]) M. Csörgő and P. Révész proved that if $W(t)$

* Research partially supported by CONICIT, Project S1-1372.

is the standard Wiener Process (i.e. $\alpha=\frac{1}{2}$, $C_0=1$) then

$$\begin{aligned}\limsup_{T\to\infty} H(T,a_T)\beta_T &= \limsup_{T\to\infty} \sup_{0\leqslant s\leqslant a_T} |W(T+s)-W(T)|\beta_T \\ &= \limsup_{T\to\infty} \sup_{0\leqslant t\leqslant T-a_T} |W(t+a_T)-W(t)|\beta_T \\ &= \limsup_{T\to\infty} |W(T+a_T)-W(T)|\beta_T = 1\end{aligned}$$

where all equalities hold with probability one and $\beta_T^{-1}=\sigma(a_T)(2\log((T\log T)/a_T))^{1/2}$. We prove in the next section that a similar result holds for Gaussian processes satisfying the conditions described above. We also consider increments of the type $I(T,a_T)$ and prove analogous results.

The methods used in the proofs are frequently similar to those in [2, 3] and many details will not be reproduced. In what follows Const denotes a constant that may vary from line to line and $\psi(x)=(1/\sqrt{2\pi}\,x)\exp(-x^2/2)$.

2. Results

All the processes considered in this section satisfy the conditions stated in Section 1. We start with two lemmas that will be needed for Theorem 1. The first is a well-known inequality due to X. Fernique [4], which is quoted without proof.

Lemma 1. *Let $\{Y(t), t\in[0,1]\}$ be a separable, real Gaussian process with*

$$E(Y(t)-Y(s))^2\leqslant \Lambda^2(|t-s|),$$

where Λ is continuous, nondecreasing in $[0,1]$ and satisfies $\int_1^\infty \Lambda(\mathrm{e}^{-u^2})\,\mathrm{d}u<\infty$, and also $EY^2(t)\leqslant \Gamma^2$. Then, whenever $x\geqslant(4\log n)^{1/2}$,

$$P\left\{\sup_{[0,1]}|Y(t)|>x\left(\Gamma+4\int_1^\infty \Lambda(n^{-u^2})\,\mathrm{d}u\right)\right\}\leqslant Cn^2\int_x^\infty \mathrm{e}^{-u^2/2}\,\mathrm{d}u$$

where C is an independent constant and $n\geqslant 2$.

Fernique's inequality is used in the proof of the next result, which is similar to an inequality of Csörgő and Révész ([3, Lemma 1]).

Lemma 2. *Let $\{X(t), t\geqslant 0\}$ be as in Section 1. Then, if $0<h\leqslant T$ and $z\geqslant 4$*

$$P\{I(T,h)>z\sigma(h)\}\leqslant \mathrm{Const}\,\frac{T}{h}\,\frac{z^{5/\alpha}}{(\log z)^{2/\alpha}}\,\psi(z).$$

Proof. Let $\delta>0$ be given, $x\geq e$ and define $N=(2/\delta)^{1/\alpha}h^{-1}$. Then

$$\begin{aligned}
&P\{I(T,h)>(1+\delta)x\sigma(h)\}\\
&\quad\leq P\left\{\max_{0\leq i<[NT]}\sup_{0<s\leq h}\left|X\left(\frac{i}{N}+s\right)-X\left(\frac{i}{N}\right)\right|>x\sigma(h)\right\}\\
&\qquad+P\left\{\max_{0\leq i\leq[NT]}\sup_{0<s\leq 1/N}\left|X\left(\frac{i}{N}+s\right)-X\left(\frac{i}{N}\right)\right|>\frac{\delta\sigma(h)}{2}x\right\}\\
&\quad\leq NT\left[P\left\{\sup_{0<s\leq h}|X(s)|>x\sigma(h)\right\}+P\left\{\sup_{0<s\leq 1/N}|X(s)|>x\sigma(1/N)\right\}\right].
\end{aligned} \tag{2}$$

We shall use Lemma 1 to obtain a bound for both probabilities. Let $\Delta>0$ and define $Y(t)=X(t\Delta)$, $t\in[0,1]$. Then $Y(t)$ is a centred, continuous Gaussian process with $E(Y(t)-Y(s))^2\leq\sigma^2(|t-s|\Delta)$ and $EY^2(t)\leq\sigma^2(\Delta)$. Therefore, if $v\geq(4\log n)^{1/2}$, we have

$$P\left\{\sup_{[0,1]}|Y(t)|\geq v\sigma(\Delta)F(n,\Delta)\right\}\leq Cn^2\int_v^\infty \mathrm{e}^{-u^2/2}\,\mathrm{d}u$$

where

$$F(n,\Delta)=1+\frac{4}{\sigma(\Delta)}\int_1^\infty\sigma(\Delta n^{-u^2})\,\mathrm{d}u\leq 1+\frac{2}{\alpha n^\alpha\log n}.$$

Therefore if $x=v(1+2/\alpha n^\alpha\log n)$ we get

$$\begin{aligned}
P\left\{\sup_{0<s\leq\Delta}|X(s)|>x\sigma(\Delta)\right\}&\leq Cn^2\int_v^\infty \mathrm{e}^{-u^2/2}\,\mathrm{d}u\\
&\leq \text{Const } n^2\psi(x)\exp\left(\frac{2\alpha x^2n^\alpha\log n+2x^2}{(2+\alpha n^\alpha\log n)^2}\right).
\end{aligned}$$

Let now $n=x^{2/\alpha}/(\log x)^{1/\alpha}$, then $v\geq(4\log n)^{1/2}$ and $2x^2/(2+\alpha n^\alpha\log n)^2\to 0$. The remaining term in the exponent converges to 1 as $x\to\infty$ and therefore

$$P\left\{\sup_{0<s\leq\Delta}|X(s)|>x\sigma(\Delta)\right\}\leq\text{Const}\frac{x^{4/\alpha}}{(\log x)^{2/\alpha}}\psi(x).$$

Using this inequality twice in (2) we obtain

$$(2)\leq\text{Const } NT\frac{x^{4/\alpha}}{(\log x)^{2/\alpha}}\psi(x)=\text{Const}\frac{T}{h\delta^{1/\alpha}}\frac{x^{4/\alpha}}{(\log x)^{2/\alpha}}\psi(x).$$

Now put $z=x(1+\delta)$, then

$$P\{I(T,h)>z\sigma(h)\}\leq\text{Const}\frac{z^{4/\alpha}}{h\delta^{1/\alpha}}\psi^{(z)}\exp\left(\frac{2\delta z+z\delta^2}{2(1+\delta)^2}\right).$$

Finally, choosing $\delta=1/z$, the proof is finished. $\square$

In the case $\alpha = \frac{1}{2}$ our inequality is far from the best possible. Recently P. Révesz has shown ([8]) that there are constants $0 < C_1 \leq C_2 < \infty$ such that

$$C_1 \leq \frac{P\{I(T, h) > \sqrt{h}\, z\}}{(T/h) z^2 \psi(z)} \leq C_2.$$

Theorem 1. $\limsup_{T\to\infty} I(T, a_T)\beta_T \leq 1$ *a.s.*

Proof. Let $T_k = \theta^k$, $\theta > 1$ and $\varepsilon > 0$. Using the previous inequality one can show that

$$\sum_{k=1}^{\infty} P\{I(T_k, a_{T_k}) \geq (1+\varepsilon)\beta_{T_k}^{-1}\} < \infty$$

and an application of Borel–Cantelli gives

$$\limsup_{k\to\infty} I(T_k, a_{T_k})\beta_{T_k} \leq 1+\varepsilon \quad \text{a.s.}$$

Suppose $T \in [T_k, T_{k+1})$, then

$$I(T, a_T)\beta_T \leq I(T_{k+1}, a_{T_{k+1}})\beta_{T_k} = I(T_{k+1}, a_{T_{k+1}})\beta_{T_{k+1}}(\beta_{T_k}/\beta_{T_{k+1}}).$$

Since $(\beta_{T_k}/\beta_{T_{k+1}})$ can be made arbitrarily close to 1 by choosing θ close to 1, and $\varepsilon > 0$ is arbitrary, the theorem follows. □

Before looking at the converse we need a couple of lemmas, the first of which is only an adaptation of the well-known lemmas of Berman [1] and Plackett and Slepian [6, 8], quoted without a proof.

Lemma 3 (Berman, Plackett, Slepian). *Let $\{X_j, j = 1, \ldots, n\}$ be centred, stationary Gaussian r.v.'s with $EX_j^2 = 1$ for all j and $EX_iX_j = r_{ij}$. Let $I_c^{+1} = [c, \infty)$ and $I_c^{-1} = (-\infty, c)$. If $c_j \in \mathbb{R}$, $j = 1, \ldots, n$ denote by F_j the event $\{X_j \in I_{c_j}^{\varepsilon_j}\}$ where ε_j is either $+1$ or -1. Let $K \subset \{1, \ldots, n\}$, then:*

(i) *$P\{\bigcap_{j\in K} F_j\}$ is an increasing function of r_{ij} if $\varepsilon_i\varepsilon_j = +1$. Otherwise it is decreasing.*

(ii) *If $\{K_l, l = 1, \ldots, s\}$ is a partition of K then*

$$\left| P\left\{\bigcap_{j\in K} F_j\right\} - \prod_{l=1}^{s} P\left\{\bigcap_{j\in K_l} F_j\right\}\right| \leq \sum_{1\leq l<m\leq s} \sum_{i\in K_l} \sum_{j\in K_m} |r_{ij}|\phi(c_i, c_j; r_{ij}^*)$$

where $\phi(x, y; r)$ is the standard bivariate Gaussian density with correlation r and r_{ij}^* is a number between 0 and r_{ij}.

The proof of the following version of the Borel–Cantelli lemma can be found in [7].

Lemma 4. *Let $\{A_n, n \geq 1\}$ be a sequence of events. If*

(i) $\sum_{1}^{\infty} P(A_n) = \infty$,

(ii) $$\liminf_{n\to\infty} \frac{\sum_{1\leq j<k\leq n}[P(A_j\cap A_k)-P(A_j)P(A_k)]}{[\sum_{j=1}^{n}P(A_j)]^2}=0,$$

then $P(A_n \text{ i.o.})=1$.

We now turn to the lower bound. Let $\rho=\lim_{T\to\infty} a_T/T$ and define an increasing sequence $(T_k, k\geq 1)$ by $T_1=1$, $T_k-a_{T_k}=T_{k-1}$ for $k>1$ if $\rho<1$. Consider the random variables defined by $Y_k=(X(T_k)-X(T_k-a_{T_k}))/\sigma(a_{T_k})$, which satisfy $EY_k=0$, $EY_k^2=1$. For $\varepsilon>0$ given let $A_k=\{Y_k>\lambda_k\}$, where $\lambda_k=(1-\varepsilon)\hat{\lambda}_k$ and $\hat{\lambda}_k=\beta_{T_k}^{-1}/\sigma(a_{T_k})$.

Theorem 2. $\limsup_{T\to\infty}|X(T)-X(T-a_T)|\beta_T\geq 1$ *a.s.*

Proof. If $\rho=1$ then necessarily $a_T=T$, $|X(T)-X(T-a_T)|=|X(T)|$ and the result was proved by Orey [5]. Suppose $\rho<1$, as in ([3], p. 33) we have that $\sum_1^\infty P(A_n)=\infty$, and we need only show that (ii) of the previous lemma holds. By Lemma 3, if $EY_jY_k\leq 0$ then $P(A_j\cap A_k)\leq P(A_j)P(A_k)$ and (ii) evidently holds; this is true if $0<\alpha\leq\frac{1}{2}$ in (1). Therefore we only have to consider the case $\frac{1}{2}<\alpha<1$. Suppose $k\geq j+3$.

$$EY_jY_k=\frac{1}{2Q^\alpha R^\alpha}(P^{2\alpha}+G(Q,R))$$

where

$$P=\sum_{j+1}^{k-1} a_{T_i},\qquad Q=a_{T_j},\qquad R=a_{T_k},$$

$$G(U,V)=(P+U+V)^{2\alpha}-(P+U)^{2\alpha}-(P+V)^{2\alpha}.$$

By Taylor's theorem,

$$G(Q,R)=-P^{2\alpha}+2\alpha(2\alpha-1)QRP^{2(\alpha-1)}+S$$

where

$$S=\frac{2\alpha(2\alpha-1)(2\alpha-2)}{3!}((Q+R)^3(P+\tau R+\tau Q)^{2\alpha-3}$$
$$-Q^3(P+\tau Q)^{2\alpha-3}-R^3(P+\tau R)^{2\alpha-3})$$

for some $0<\tau<1$. It is easy to see that

$$S\leq\frac{2\alpha(2\alpha-1)(2-2\alpha)}{3!}Q^3(P+\tau Q)^{2\alpha-3}\quad\text{and}$$

$$G(Q,R)\leq -P^{2\alpha}+3\alpha(2\alpha-1)QRP^{2\alpha-2}$$

whence

$$EY_kY_j\leq \mathrm{Const}(a_{T_j}a_{T_k})^{1-\alpha}\left(\sum_{j+1}^{k-1}a_{T_i}\right)^{2(\alpha-1)}. \tag{3}$$

Since $\rho < 1$ we may assume, without loss of generality, that $a_1 < 1$, and then $T_k(1-a_1) \leq T_{k-1}$ which implies $a_{T_k} \leq (1-a_1)^{-1} a_{T_{k-1}}$ and

$$EY_k Y_j \leq \text{Const}\left(a_{T_j} \Big/ \sum_{j+1}^{k-1} a_{T_i} \right)^{1-\alpha} \leq \text{Const}(k-j-1)^{\alpha-1} \equiv \eta_{jk}$$

as long as $k \geq j+3$.

We can now turn to the proof of (ii). Lemma 3 shows that $P(A_j \cap A_k) - P(A_j)P(A_k) \leq r_{jk}\phi(\lambda_j, \lambda_k; r^*_{jk})$ with $r_{jk} = EY_jY_k$. For our purpose it is enough to consider, for some fixed m,

$$\sum_{k=m}^{n} \sum_{j=1}^{k-3} (P(A_j \cap A_k) - P(A_j)P(A_k))$$

$$\leq \left(\sum_{k=m}^{n} \sum_{j=1}^{k-\gamma_k} + \sum_{k=m}^{n} \sum_{j=k-\gamma_k}^{k-3} \right) r_{jk}\phi(\lambda_j, \lambda_k; r^*_{jk})$$

where $\gamma_k = \lambda_k^{4/(1-\alpha)}$. We start with

$$\sum_{k=m}^{n} \sum_{j=1}^{k-\gamma_k} \frac{r_{jk}}{2\pi(1-r^{*2}_{jk})^{1/2}} \exp\left\{ -\frac{\lambda_j^2 + \lambda_k^2 - 2\lambda_j\lambda_k r^*_{jk}}{2(1-r^{*2}_{jk})} \right\}$$

$$\leq \sum_{k=m}^{n} \sum_{j=1}^{k-\gamma_k} \frac{\eta_{jk}\lambda_k\lambda_j}{(1-\eta^2_{jk})^{1/2}} \psi(\lambda_j)\psi(\lambda_k)\exp\left\{ \frac{r^{*2}_{jk}(\lambda_j^2 + \lambda_k^2) - 2\lambda_j\lambda_k r^*_{jk}}{2(1-r^{*2}_{jk})} \right\}$$

$$\leq \sum_{k=m}^{n} \sum_{j=1}^{k-\gamma_k} \frac{\eta_{jk}\lambda_k^2}{(1-\eta^2_{jk})^{1/2}} \psi(\lambda_j)\psi(\lambda_k)\exp\{\eta_{jk}\lambda_k^2\}$$

but $\eta_{jk}\lambda_k^2 \leq \text{Const}\, \gamma_k^{\alpha-1}\lambda_k^2 \leq \text{Const}\, \lambda_k^{-2}$. Let $\delta > 0$, then, choosing m appropriately (but fixed) the sum we are considering is

$$\leq \delta \sum_{k=m}^{n} \sum_{j=1}^{k-\gamma_k} P(A_i)P(A_j) \leq \delta\left(\sum_{k=1}^{n} P(A_k) \right)^2. \tag{4}$$

The second sum is

$$\leq \sum_{k=m}^{n} \sum_{j=k-\gamma_k}^{k-1} \frac{r_{jk}\lambda_j}{\sqrt{2\pi}\,(1-r^{*2}_{jk})^{1/2}} \psi(\lambda_j)\exp\left\{ -\frac{(\lambda_k - r^*_{jk}\lambda_j)^2}{2(1-r^{*2}_{jk})} \right\}$$

$$\leq \sum_{k=m}^{n} \sum_{j=k-\gamma_k}^{k-1} \frac{\lambda_j}{(1-r^2)^{1/2}} \psi(\lambda_j)\exp\left\{ -\frac{\lambda_k^2}{2}\left(\frac{1-r}{1+r}\right) \right\}$$

where $r < 1$ is the maximum of the convariances r_{jk}. If $B = (1-r)/2(1+r)$ we have

$$\leq \sum_{k=m}^{n} \sum_{j=k-\gamma_k}^{k-1} \frac{\lambda_j\psi(\lambda_j)}{(1-r^2)^{1/2}} \exp\{-B\lambda_k^2\}.$$

Let us consider first the sum over the indices k in the set $A=\{k: m\leqslant k\leqslant n$ and $\lambda_k\geqslant((2/B)\log k)^{1/2}\}$. Then

$$\sum_{k\in A}\sum_{j=k-\gamma_k}^{k-1}\frac{\lambda_j\psi(\lambda_j)}{(1-r^2)^{1/2}}\exp\{-B\lambda_k^2\}$$

$$\leqslant \mathrm{Const}\sum_{k\in A}\frac{(\log k)^{1/2}}{k^2}\sum_{j=1}^{n}P(A_j)\leqslant \mathrm{Const}\sum_{j=1}^{n}P(A_j). \tag{5}$$

If $k\in A'=\{k: m\leqslant k\leqslant n$ and $\lambda_k<((2/B)\log k)^{1/2}\}$ then $\gamma_k<((2/B)\log k)^{2/1-\alpha}$ and if $j=k-\gamma_k$ then, for some $D>0$, $k<j+D(\log j)^{2/1-\alpha}=j+\xi_j$ say. Changing the order of summation,

$$\sum_{k\in A'}\sum_{j=k-\gamma_k}^{k-1}\frac{\lambda_j\psi(\lambda_j)}{(1-r^2)^{1/2}}\exp\{-B\lambda_k^2\}$$

$$\leqslant \mathrm{Const}\sum_{j=m-\gamma_m}^{n-1}\sum_{k=j+1}^{j+\xi_j}\lambda_j\psi(\lambda_j)\exp\{-B\lambda_k^2\}$$

$$\leqslant \mathrm{Const}\sum_{j=m-\gamma_m}^{n-1}\xi_j\lambda_j\psi(\lambda_j)\exp\{-B\lambda_j^2\}\leqslant \mathrm{Const}\sum_{j=1}^{n}P(A_j). \tag{6}$$

Therefore using (4), (5) and (6), given δ there is an m such that

$$\sum_{k=m}^{n}\sum_{j=1}^{k-3}(P(A_j\cap A_k)-P(A_j)P(A_k))\leqslant\delta\left(\sum_{j=1}^{n}P(A_j)\right)^2+\mathrm{Const}\sum_{j=1}^{n}P(A_j)$$

and this is enough to prove (ii) in the case $\rho<1$. □

As an immediate consequence we have

Theorem 3. *If $\{X(t), t\geqslant 0\}$ is a Gaussian process satisfying the conditions of the first section,*

$$\limsup_{T\to\infty} I(T,a_T)\beta_T=\limsup_{T\to\infty} H(T,a_T)\beta_T$$

$$=\limsup_{T\to\infty}|X(T+a_T)-X(T)|\beta_T=1$$

where all equalities hold with probability one.

In the next theorem we show that, as in the case of the Wiener process, under certain additional conditions the limsup can be replaced by a lim.

Theorem 4. *If* $\lim_{T\to\infty}(\log T/a_T)/(\log\log T) = \infty$ *then*

$$\lim_{T\to\infty} I(T, a_T)\beta_T = \lim_{T\to\infty} H(T, a_T)\beta_T$$
$$= \lim_{T\to\infty} \sup_{0\le t\le T-a_T} |W(t+a_T) - W(t)|\beta_T = 1$$

where all equalities hold with probability one.

Proof. Let $C(T) = \sup_{0\le t\le T-a_T} |X(t+a_T) - X(t)|\beta_T$. We shall prove that $\liminf_{T\to\infty} C(T) \ge 1$ a.s. For some $\delta > 0$ define $T_n = (1+\delta)^n$, $\zeta(T) = [T/a_T] - 1$ and

$$V(k, n) = (X((k+1)a_{T_n}) - X(ka_{T_n}))/\sigma(a_{T_n})$$

for $0 \le k \le \zeta(T_n)$, $n \ge 1$. The variables $V(k, n)$ have $EV(k, n) = 0$, $EV^2(k, n) = 1$ for all (k, n), and it can be shown as in the proof of theorem 2 that if $k \ge j+1$, $r_n(k, j) = EV(k, n)V(j, n) \le \text{Const}(k-j)^{2(\alpha-1)}$ and is negative if $\alpha \le \frac{1}{2}$. Using Lemma 3, if $\alpha \le \frac{1}{2}$,

$$P\left\{\max_{0\le k\le \zeta(T_n)} |V(k, n)| \le \lambda_n\right\} \le \prod_{k=0}^{\zeta(T_n)} P\{|V(k, n)| \le \lambda_n\}$$

and one can argue as in [3] to show that

$$\sum_{n=1}^{\infty} P\left\{\max_{0\le k\le \zeta(T_n)} |V(k, n)| \le \lambda_n\right\} < \infty.$$

If $\alpha > \frac{1}{2}$ we have to consider an extra term coming from (ii) of Lemma 3. Suppose that ε is small enough so that $\varepsilon < 1-\alpha$ and $2(1-\varepsilon)^2 > 1+r$ where $r = \max\{r_n(k, j): n \ge 1, 1 \le k < j \le \zeta(T_n)\}$. Then

$$\sum_{n=1}^{\infty} \sum_{k=0}^{\zeta(T_n)} \sum_{j=k+1}^{\zeta(T_n)} r_n(k, j)\phi(\lambda_n, \lambda_n; r_n^*(k, j))$$
$$= \sum_{n=1}^{\infty} \sum_{k=0}^{\zeta(T_n)} \left(\sum_{j=k+1}^{k+\mu_n-1} + \sum_{j=k+\mu_n}^{\zeta(T_n)}\right) \frac{r_n(k, j)}{(1-r_n^2(k, j))^{1/2}} \exp\left\{-\frac{\lambda_n^2}{1+r_n^*(k, j)}\right\}$$

where $\mu_n = \lambda_n^{1/(1-\alpha)}$. Let $\nu > 0$ be defined by $1+\nu = 2(1-\varepsilon)^2/1+r$, then the first sum is

$$\le \text{Const} \sum_{n=1}^{\infty} \sum_{k=0}^{\zeta(T_n)} \sum_{j=k+1}^{k+\mu_n} \exp\left\{-\frac{(1+\nu)\hat{\lambda}_n^2}{2}\right\}$$
$$\le \text{Const} \sum_{n=1}^{\infty} \frac{T_n}{a_{T_n}} \mu_n \left(\frac{a_{T_n}}{T_n \log T_n}\right)^{1+\nu} < \infty.$$

The second sum can be bounded by

$$\text{Const}\sum_{n=1}^{\infty}\sum_{k=0}^{\zeta(T_n)}\sum_{j=k+\mu_n}^{\zeta(T_n)}\frac{1}{(j-k)^{2(1-\alpha)}}\exp\left\{-\lambda_n^2+\frac{\text{Const}\,\lambda_n^2}{\mu_n^{2(1-\alpha)}}\right\}$$

$$\leq \text{Const}\sum_{n=1}^{\infty}\sum_{k=0}^{\zeta(T_n)}\left(\frac{a_{T_n}}{T_n\log T_n}\right)^{2(1-\varepsilon)}\sum_{j=k+\mu_n}^{\zeta(T_n)}\frac{1}{(j-k)^{2(1-\alpha)}}$$

$$\leq \text{Const}\sum_{n=1}^{\infty}\zeta^{2\alpha}(T_n)\left(\frac{a_{T_n}}{T_n\log T_n}\right)^{2(1-\varepsilon)}<\infty$$

and by Borel–Cantelli we have that

$$\liminf_{n\to\infty}\max_{0\leq k\leq\zeta(T_n)}|V(k,n)|\lambda_n^{-1}\geq 1 \text{ a.s.}$$

Suppose now $T_n\leq T<T_{n+1}$, then $0\leq a_T-a_{T_n}\leq\delta a_T$ and therefore

$$\begin{aligned}C(T)&=\sup_{0\leq t\leq T-a_T}|X(t+a_T)-X(t)|\beta_T\\&\geq\max_{0\leq k\leq\zeta(T_n)}|V(k,n)|\sigma(a_{T_n})\beta_{T_{n+1}}\\&\quad-\sup_{0\leq t\leq T-\delta a_T}\ \sup_{0\leq s\leq\delta a_T}|X(t+s)-X(t)|\beta_T\end{aligned}$$

and using Theorem 1 and the fact that $\beta_{T_{n+1}}/\beta_{T_n}$ is arbitrarily close to one if δ is small enough completes the proof. □

Acknowledgement

The author would like to thank W. Urbina and M. Wschebor for many interesting conversations on the topic of this paper, and also the referees for pointing out some errors in a previous version, as well as for several comments that have led to improvements in this work.

References

[1] S.M. Berman, Limit theorems for the maximum term in stationary sequences, Ann. Math. Statist. 35 (1964) 502–516.

[2] M. Csörgő and P. Révész, How big are the increments of a Wiener process?, Ann. Probab. 7 (1979) 731–733.

[3] M. Csörgő and P. Révész, Strong approximations in Probability and Statistics (Academic Press, New York, 1981).

[4] X. Fernique, Continuité des processus Gaussiens, C.R. Acad. Sci. Paris 258 (1964) 6058–6060.

[5] S. Orey, Growth rate of certain Gaussian processes, in: Proc. Sixth Berkeley Symposium Math. Statist. Prob. Vol. 2 (University of California Press, Berkeley and Los Angeles, 1971) pp. 443–451.
[6] R.L. Plackett, A reduction formula for Normal multivariate integrals, Biometrika 41 (1954) 351–360.
[7] A. Renyi, Probability Theory (North-Holland, Amsterdam 1970).
[8] P. Révész, On the increments of Wiener and related processes. Ann. Probab. 10 (1982) 613–627.
[9] D. Slepian, The one-sided barrier problem for Gaussian noise, Bell System Tech. J. 41 (1962) 463–501.